Last time:

$K/\mathbb{Q}$ finite, $0 \neq I \subseteq K$ fract. ideal,

$\sigma_1, \dots, \sigma_{r_1}, \sigma_{r_1+1}, \dots, \sigma_{r_1+2r_2}$,

$$\overline{\sigma_{r_1+2j}} = \sigma_{r_1+2j-1}, \; j = 1, \dots, r_2$$

$n = r_1 + 2r_2$

Thm: 1) Given $c_1, \dots, c_{r_1+r_2} > 0$

with $\prod_{i=1}^{r_1+r_2} c_i > \left(\frac{2}{\pi}\right)^{r_2} \cdot \sqrt{|\Delta_K|} \cdot N(I)$

$\Rightarrow \exists \alpha \in I \setminus \{0\}$ with

$$|\sigma_i(\alpha)| < c_i, \; i = 1, \dots, r_1$$

$$|\sigma_{r_1+2j}(\alpha)|^2 < c_{r_1+j}, \; j = 1, \dots, r_2$$

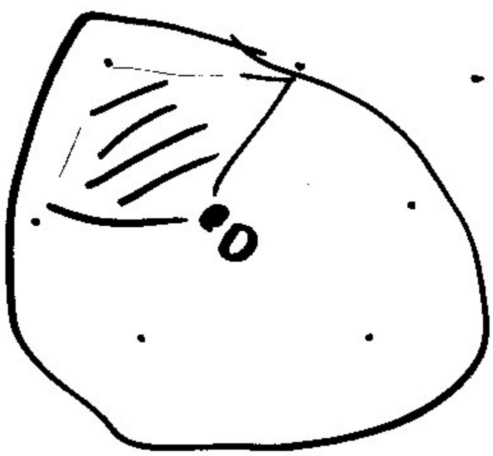
2) ex. $\alpha \in I \setminus \{0\}$, s.t.

$$|N_{K/\mathbb{Q}}(\alpha)| \le \left(\frac{4}{\pi}\right)^{r_2} \cdot \frac{n!}{n^n} \sqrt{|\Delta_K|} \cdot N(I)$$

(Applications of Minkowski's la:

$\iota : K \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong \mathbb{R}^n \supseteq \iota(I)$ lattice

$(\sigma_1, \dots, \sigma_{r_1}, \sigma_{r_1+2}, \dots, \sigma_{r_1+2r_2})$



$$\mathrm{Vol}\left(\mathbb{R}^n / \iota(I)\right) = \frac{1}{2^{r_2}} \cdot \sqrt{|\Delta_K|} \cdot N(I)$$

)

Today: Dirichlet's unit thm

$U_K := \mathcal{O}_K^*$

Thm: $\exists$ s.e.s. $0 \to W_K \to U_K \to U_K/W_K \to 0$

with

* W_K finite cyclic,

  $W_K = \mu_\infty(K) := \{ x \in K \mid \exists n > 0 : x^n = 1 \}$

* U_K / W_K fin. gen., free ab. group of rk $r_1 + r_2 - 1$

Ex: * $K/\mathbb{Q}$ imag. quadr. $\Rightarrow U_K = W_K$

* $K = \mathbb{Q}(\sqrt{5}) \Rightarrow U_K / W_K = \langle \frac{1+\sqrt{5}}{2} \rangle$

Lemma: $u \in \mathcal{O}_K$

Then $u \in W_K \Leftrightarrow |\sigma_i(u)| = 1 \ \forall i = 1, \dots, n$

Proof: "$\Rightarrow$" ✓

"$\Leftarrow$" $u \in \mathcal{O}_K^\times = U_K$ (bec. $N_{K/\mathbb{Q}}(u) = \pm 1$)

Note the set of such $u \in \mathcal{O}_K$ forms a subgrp. of U_K
U

The coeff. of the min. poly. of elts in U are bdd (in terms of ~~A~~ n)

$\Rightarrow U$ finite $\Rightarrow U \subseteq W_K$

Set $\ell : U_K \xrightarrow{\lambda} (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \xrightarrow{\log} \mathbb{R}^{r_1+r_2}$

$$(x_1, \dots, x_{r_1}, z_1, \dots, z_{r_2}) \longmapsto (\log|x_i|, \log|z_j|^2)$$

$$i = 1, \dots, r_1,\ j = 1, \dots, r_2$$

$\Rightarrow \ell$ grp. hom & $\ker \ell = W_K$

$\operatorname{Im} \ell \subseteq H := \{ y_1, \dots, y_{r_1+r_2} \in \mathbb{R}^{r_1+r_2} \mid \sum_{i=1}^{n} y_i = 0 \}$

$(\ \forall\, u \in U_K,\ N_{K/\mathbb{Q}}(u) = \pm 1)$

Claim: $\ell(U_K)$ is a lattice in H

$(\Leftrightarrow \ell(U_K) \simeq \mathbb{Z}^{r_1+r_2-1})$

Discreteness of $\ell(U_K)$ in H

For $\delta > 0$ set
$$B_\delta := \{(y_1,\dots,y_{r_1}, z_1,\dots,z_{r_2}) \in \mathbb{R}^{r_1}\times\mathbb{C}^{r_2} \mid e^{-\delta}\le |y_i| \le e^{\delta},\ e^{-\delta}\le |z_j|^2\le e^{\delta}\}$$

$\Rightarrow$ $B_\delta \cap \iota(\mathcal{O}_K) = \iota(W_K)$ for δ suff. small as $\iota(\mathcal{O}_K)\subseteq \mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$ discrete

Consider
$$C_\delta := \{(x_1,\dots,x_{r_1+r_2})\in\mathbb{R}^{r_1+r_2} \mid |x_i|\le\delta\}$$

$$\Rightarrow \ell(U_K)\cap C_\delta \subseteq \underset{\text{Log}}{\ell}(\iota(\mathcal{O}_K)\cap B_\delta) = \ell(W_K) = \{0\} \quad (\delta \text{ suff. small})$$

$\Rightarrow \ell(U_K) \leq KH$ discrete

Rk $\ell(U_K) = r_1 + r_2 - 1$

La: $1 \leq k \leq r_1 + 2r_2$

$\Rightarrow \exists\, u_k \in U_K$, s.t.

$|\sigma_k(u_k)| > 1,\ |\sigma_i(u_k)| < 1 \quad \forall \sigma_i \neq \sigma_k$

$\& \ \sigma_i \neq \bar{\sigma}_k$

Proof: Rename σ_i,

$\sigma_1, \dots, \sigma_{r_1}, \sigma_{r_1+1}, \dots, \sigma_{r_1+2r_2}$,

s.t. $\bar{\sigma}_{r_1+j} = \sigma_{r_1+r_2+j}$

Fix $A > \left(\frac{2}{\pi}\right)^{r_2} \cdot \sqrt{|\Delta_K|}$, $c_1, \dots, c_{r_1+r_2} > 0$,

s.t. $c_i < 1$ for $1 \leq i \leq r_1 + r_2$, $i \neq k$,

and $c_k = A \Big/ \prod_{i \neq k} c_i$ (wlog $k \in \{1, \dots, r_1 + r_2\}$)

$\Rightarrow$ Then exists $a_1 \in \mathcal{O}_K \setminus \{0\}$, s.t.

$$|\sigma_i(a)| < c_i \ , \ i = 1, \dots, r_1$$

$$|\sigma_j(a)|^2 < c_j \ , \ j = 1, \dots, r_2$$

Set
$$\left.\begin{aligned} c_i^{(1)} &:= |\sigma_i(a)| \ , \ i = 1, \dots, r_1 \\ c_j^{(1)} &:= |\sigma_j(a)|^2 \ , \ j = 1, \dots, r_2 \end{aligned}\right\} i \neq k$$

and $c_k^{(1)} = A \Big/ \prod\limits_{i \neq k} c_i^{(1)}$

$\Rightarrow$ Get $a_2 \in \mathcal{O}_K \setminus \{0\}$, s.t.

$$|\sigma_i(a_2)| < |\sigma_i(a_1)| (\leq 1|,$$

$$\cancel{} \ i = 1, \dots, r_1 + r_2, \ i \neq k$$

and $|N_{K/\mathbb{Q}}(a_2)| = \prod\limits_{i=1}^{r_1} |\sigma_i(a)| \cdot \prod\limits_{j=1}^{r_2} |\sigma_j(a)|^2$

$\vdash$

$$< \prod_{i=1}^{r_1+r_2} c_i^{(1)} = A$$

$\Rightarrow$ Iteratively, get sequence

$a_1, a_2, \ldots\ldots \in \mathcal{O}_K \setminus \{0\}$

$|\sigma_i(a_{n+1})| < |\sigma_i(a_n)|, \quad i \neq k$

$|N_{K/\mathbb{Q}}(a_n)| < A$

But $\{\mathfrak{a} \subseteq \mathcal{O}_K \mid N(\mathfrak{a}) \leq A\}$ is finite

$\Rightarrow (a_n) = (a_m)$ for some $n < m$.

Set $u_k = \frac{a_m}{a_n} \in U_K$

$(N_{K/\mathbb{Q}}(u_k) = \pm 1 \Rightarrow |\sigma_k(u_k)| > 1 \,\})$

Consider $u_1,\dots,u_{r_1+r_2}$ from the la

Consider matrix

$$B := (\ell(u_1),\dots,\ell(u_{r_1+r_2}))$$

$\Rightarrow$ $B_{ii}>0$, $B_{ij}<0\ \forall i\neq j$, $\sum_{i=1}^{r_1+r_2} B_{ij}=0\ \forall j$

$\Rightarrow$ B has ~~r~~ rk r_1+r_2-1

Indeed, set $m=r_1+r_2$.

Show: First $m-1$ rows of B are l.u.

Assume $\sum_{i=1}^{m-1} x_i\cdot B_{ij}=0\ \forall j=1,\dots,m$

$(x_1,\dots,x_{m-1})\neq 0$

$j=m \Rightarrow$ not all of x_i are equal

$$\begin{pmatrix} >0 & & <0 \\ & \ddots & \\ <0 & & >0 \end{pmatrix}$$

j_0

Let $j_0 = 1, \ldots, m-1$, s.t.

$$x_{j_0} = \max_{1 \le j \le m-1} \{x_j\} \geq 0$$

$$\Rightarrow 0 = \sum_{i=1}^{m-1} x_i B_{ij_0}$$

$$= \underbrace{x_{j_0}}_{\geq 0} \cdot \underbrace{\sum_{i=1}^{m-1} B_{ij_0}}_{\geq 0} + \underbrace{\sum_{\substack{i=1 \\ i \neq j_0}}^{m-1} \overbrace{(x_i - x_{j_0})}^{<0} \cdot \overbrace{B_{ij_0}}^{<0}}_{>0}$$

$$> 0 \quad \text{↯}$$

$\square$

Def: $K/\mathbb{Q}$ finite

$$R_K := \operatorname{Vol}(H / l(U_K)) \quad \text{regulator}$$

$$= |\det(n, l(u_1), \ldots, l(u_{r_1+r_2-1}))|$$

with $n = \frac{1}{r_1+r_2}(1, \ldots, 1)$

$u_1, \dots, u_{r_1+r_2-1}$ syst. of fund. units
i.e. images in $U_K/_{W_K}$ is basis.

6.1. Distributions of ideals in number fields

$K/_{\mathbb{Q}}$ finite, $n := [K:\mathbb{Q}]$

Recall: $t \geq 1$

$$N(t) := \#\{0 \neq I \subseteq O_K \mid N(I) \leq t\}$$
is finite

Aim:
$$N(t) = \frac{2^{r_1} \cdot (2\pi)^{r_2} \cdot R_K \cdot h_K}{w \cdot \sqrt{|\Delta_K|}} t + O\left(t^{1-1/n}\right)$$

Here: $w = \# W_K$

(Recall: $f(t) = g(t) + O(t^{1-1/n})$

means that there exists $A > 0$, s.t.

$$|f(t) - g(t)| \leq A \cdot t^{1-\frac{1}{n}} \text{ for } t \geq 1).$$

Ex: $K = \mathbb{Q} \Rightarrow r_1 = 1, r_2 = 0, R_K := 1, h_K = 1,$

$$w = 2, \ \Delta_K = 1$$

$$\Rightarrow \ RHS = t + O(1)$$

$$LHS = N(t) = N(\lfloor t \rfloor) = \lfloor t \rfloor$$

$$\Rightarrow |t - \lfloor t \rfloor| \leq 1$$

In general, fix $C \in Cl_K$

set $N_C(t) := \#\{0 \neq I \subseteq O_K \mid I \in C, N(I) \leq t\}$, $t \geq 1$

& show

$$N_C(t) = \frac{2^{r_1}(2\pi)^{r_2} \cdot R_K}{w \cdot \sqrt{|\Delta_K|}} \cdot t + O(t^{1-\frac{1}{n}})$$

$$\left(N(t) = \sum_{C \in Cl_K} N_C(t) \right)$$

La: Let $J \in C^{-1}$, $t \geq 1$

Set $S_t := \{x \in J \setminus \{0\} \mid |N_{K/\mathbb{Q}}(x)| \leq t \cdot N(J)\}/_{\sim}$

$x \sim u \cdot x$, $\forall u \in U_K$

$$\Rightarrow \quad S_t \xrightarrow{1:1} \{0 \neq I \subseteq \mathcal{O}_K \mid I \in C,\ N(I) \leq t\}$$

$$\alpha \longmapsto (\alpha) \cdot J^{-1}$$

Proof: $\alpha \in S_t$

$\Rightarrow I := (\alpha) \cdot J^{-1} \subseteq J \cdot J^{-1} = \mathcal{O}_K$

$N(I) \leq t$

Conv., $I \in C = [J^{-1}]$

$\Rightarrow$ ex. $\alpha \in K$, s.t. $(\alpha) \cdot J^{-1} = I \subseteq \mathcal{O}_K$

$\Rightarrow \alpha \in J$ & $N_{K/\mathbb{Q}}(\alpha) \cdot N(J)^{-1} = N(I) \leq t$

∎

⚠ Counting elements in S_t is (roughly) counting lattice pts in some region with $(\iota(J))$ bdd norm.

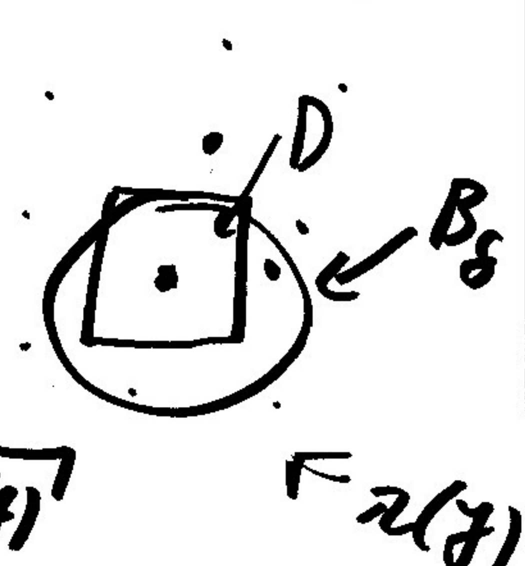
Ex: $K/\mathbb{Q}$ imag. quadr.

Fix $\iota: K \hookrightarrow \mathbb{C}^{H} \simeq \mathbb{R}^2$,

Recall $\mathrm{vol}(\mathbb{R}^2/\iota(J)) = \frac{1}{2}\sqrt{|\Delta_K|}\cdot N(J)$

Set $B_\delta := \{ z \in \mathbb{C} \mid |z| \le \delta \}$

Note: $|N_{K/\mathbb{Q}}(x)| = |\iota(x)|^2$



$\Rightarrow N_C(t) = \frac{\#\,\iota(J) \cap B_{\sqrt{t\cdot N(J)}}}{w}$

Set $n(t) = \#(\iota(J) \cap B_{\sqrt{t N(J)}})$

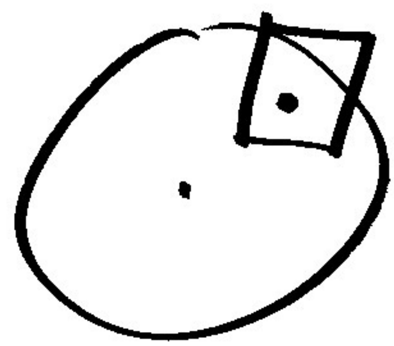
Set $D := \sum_{i=1}^{2} \left[-\frac{1}{2}, \frac{1}{2}\right]\alpha_i$, $\langle \alpha_1, \alpha_2 \rangle_{\mathbb{Z}} = \mathfrak{J}$

$$n_+(t) = \#\{\alpha \in \mathfrak{J} \mid \alpha + D \cap B_{\sqrt{t \cdot N(\mathfrak{J})}} \neq \emptyset\}$$

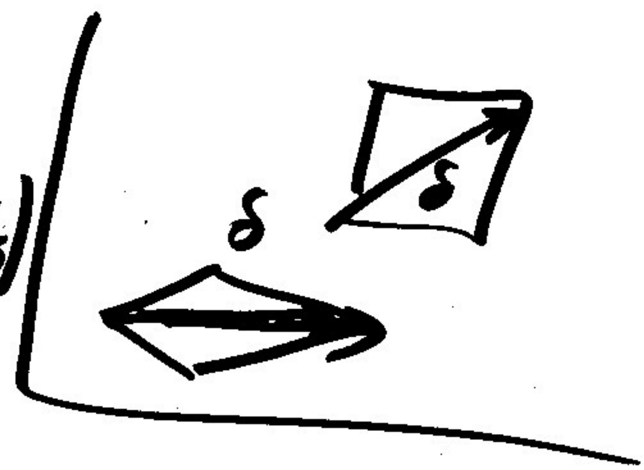
$$\begin{array}{l} \geq \\ n(t) \\ \geq \end{array}$$

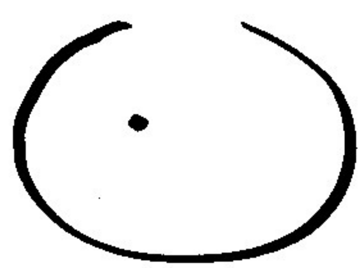
$$n_-(t) = \#\{\alpha \in \mathfrak{J} \mid \alpha + D \subseteq B_{\sqrt{t \cdot N(\mathfrak{J})}}\}$$

Let $\delta := \max\{|z-y| \mid z, y \in D\}$



$$\Rightarrow \mu(D) \cdot n_+(t) \leq \mu(B_{\sqrt{t \cdot N(\mathfrak{J})} + \delta})$$

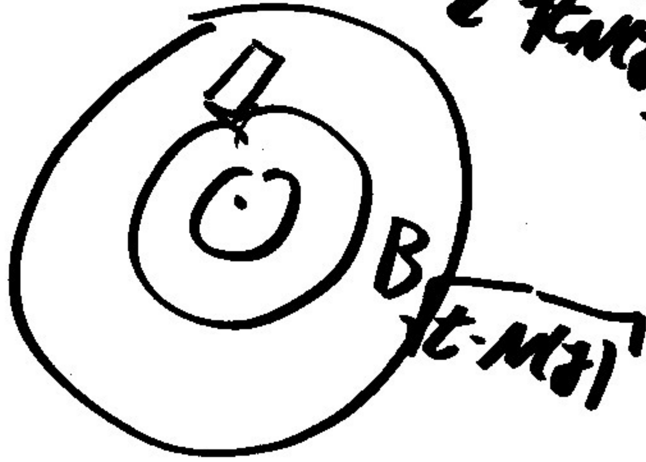




$\leq \mu(B_{\sqrt{tN(J)}+\delta})$

$\leq \mu(D)\cdot n(t)$

$\geq \mu(B_{\sqrt{tN(J)}}-\delta)$



$$\Rightarrow \frac{\mu(B_{\sqrt{tN(J)}}-\delta)}{\mu(D)} \leq n(t) \leq \frac{\mu(B_{\sqrt{tN(J)}}+\delta)}{\mu(D)}$$

$$\parallel$$

$$\frac{2\pi(\sqrt{t\cdot N(J)}-\delta)^2}{N(J)\cdot\sqrt{|\Delta_n|}} \leq n(t) \leq \frac{2\pi(\sqrt{t\cdot N(J)}+\delta)^2}{N(J)\cdot\sqrt{|\Delta_n|}}$$

$$\Rightarrow \left| n(t) - \frac{2\pi\cdot t}{\sqrt{|\Delta_n|}} \right| \leq A\cdot t^{\frac{1}{2}}$$

$$\Rightarrow N_c(t) = \frac{2\pi}{w\sqrt{|\Delta_n|}}\cdot t + O(At^{\frac{1}{2}}) \quad \square$$